

A Multidimensional Tropical Optimization Problem with Nonlinear Objective Function and Linear Constraints

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Abstract

We examine a multidimensional optimization problem in the tropical mathematics setting. The problem is to minimize a nonlinear function defined on a finite-dimensional semimodule over an idempotent semifield under linear inequality constraints. Given two arbitrary matrices, the objective function is defined by one of the matrices through the use of multiplicative conjugate transposition, whereas the inequality constraints are given by the other matrix. We start with an overview of known tropical optimization problems with both linear and nonlinear objective functions, subject to linear equality and inequality constraints. A short introduction to tropical algebra is given to provide a formal framework to solve the optimization problem under study. As a preliminary result, a general solution to a linear inequality with an arbitrary matrix is presented. We describe an example optimization problem drawn from project scheduling and then offer a general representation of the problem. Furthermore, we introduce an additional unknown variable and reduce the problem to solving a linear inequality, where the new variable plays the role of a parameter. A necessary and sufficient condition for the inequality to hold is used to evaluate the parameter, whereas the solution of the inequality is taken as a solution for the optimization problem. As a result, under fairly general conditions, a complete direct solution to the problem is derived in a compact vector form. Numerical and graphical examples for two-dimensional problems are given to illustrate obtained solutions.

Key-Words: idempotent semifield, multidimensional optimization problem, nonlinear objective function, linear inequality constraints, project scheduling.

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1 Introduction

Tropical (idempotent) mathematics, as an applied mathematical theory and methods based on the notion of idempotent semirings, has its origin in a few works, including [1, 2, 3, 4], all inspired by problems from operations research. Over the last decades, there have been many theoretical results achieved and various applications developed in a number of studies, which are reported in such monographs as [5, 6, 7, 8, 9, 10, 11, 12, 13, 14], as well as in a wide range of contributed papers.

Even though models and methods of tropical mathematics find expanding applications in many fields, including engineering, computer science, and economics, the scope of operations research remains to be the main domain, which supplies more new problems to solve and uses more new results to apply. One of the areas of investigation directly concerned with problems in operations research is tropical optimization, which deals with analysis and solution of optimization problems formulated in the tropical mathematics setting.

A minimax earliness problem in machine scheduling in [15] is an early instance of real-life problems that can be represented and solved within the framework of tropical mathematics. Further examples include multidimensional optimization problems appeared in location analysis, transportation networks, decision making, and discrete event systems.

The optimization problems usually take the form of minimizing (maximizing) functions defined on finite-dimensional semimodules over idempotent semifields, and may have additional equality and inequality constraints imposed on the feasible solutions. Some problems involve both an objective function and constraints that are linear in the tropical mathematics sense. These tropical linear optimization problems, which appear to be formal analogues of those in conventional linear programming, were apparently first considered in [16, 17, 7].

In another important class of problems dating back in the literature to the works [15, 7], the constraints are still linear, whereas the objective function is not. The class actually combines problems with functions involving a conjugate transposition operator, which does not preserve tropical linearity.

There are a number of solution techniques developed to handle the tropical optimization problems. Some problems have complete solutions obtained in an explicit vector form in terms of a general semiring as those in the early works [16, 15, 17, 7]. For other problems, existing solutions are given for the case of a particular semifield in the form of iterative algorithms that give particular solutions if any, and indicate that there is no solution otherwise (see, for instance, [18, 19]).

In this paper, we examine a multidimensional optimization problem that appears, in particular, in project management when an optimal schedule is constructed to minimize flow time of activities in a project under various

activity precedence constraints.

We formulate the problem in terms of a general semimodule over an idempotent linearly ordered radicable semifield. Given two arbitrary matrices, the problem is to minimize a nonlinear objective function defined by one of the matrices through multiplicative conjugate transpose, and involves linear inequality constraints given by the other matrix. The problem presents a sufficient extension of that considered in [20], where an irreducibility condition is imposed on the matrices, whereas the inequality constraints are less general. Solutions to an unconstrained version of the problem together with related applications in location analysis and stochastic discrete event systems are presented in [21, 22, 13, 23, 24].

To solve the problem, we implement and further develop results offered in [25, 22, 13], including solutions to linear inequalities and extremal properties of eigenvalues of matrices. We follow the approach proposed in [20], which introduces an additional unknown variable and then reduces the problem to solving a linear inequality, where the new variable plays the role of a parameter. A necessary and sufficient condition for the inequality to hold is used to evaluate the parameter, whereas the solution of the inequality is taken as a solution of the initial problem. As a result, under fairly general conditions, a complete direct solution to the problem is derived in a compact vector form, which is suitable to both further analysis and applications.

The rest of the paper is organized as follows. We start with an overview of some known multidimensional tropical optimization problems with both linear and nonlinear objective functions and linear constraints. We briefly discuss related solution methods and indicate existing areas of application.

A short introduction to tropical algebra is then given to provide an appropriate formal framework in terms of a general idempotent semifield to solve the optimization problem under study. The introduction includes clear graphical illustrations intended to clarify main definitions and basic facts by the example of the classical semifield $\mathbb{R}_{\max,+}$.

Furthermore, a complete explicit solution for a linear inequality is obtained as an important prerequisite. The solution extends known results for the inequality with an irreducible matrix to the case of irreducible matrices.

Finally, we offer an example problem drawn from project scheduling and then give a formal description of the optimization problem of interest. We exploit the above solution for inequalities to obtain our main result, a complete solution to the problem. Numerical examples of solving two-dimensional problems and related graphical illustrations are also included.

2 Tropical optimization problems

Multidimensional tropical optimization problems constitute a significant research domain in tropical mathematics, dating back to the works [16, 15].

The problems arise in real-world applications in various areas. Specifically, [15, 7, 19, 26, 27, 28, 29] provide topical solutions to problems in project scheduling. Furthermore, [30, 23, 24, 31] solve problems in location analysis. Finally, there are applications developed in transportation networks [7, 26], in decision making [32], and in discrete event system analysis [21].

Below we give a short overview of one class of multidimensional optimization problems and briefly discuss their solution methods. The problems are formulated as to minimize functions defined on vectors of a finite-dimensional semimodule over an idempotent linearly ordered radicable semifield. Both unconstrained and constrained problems are considered, where the constraints have the form of linear vector equations and inequalities.

We present optimization problems with both linear and nonlinear objective functions. However, only those nonlinear functions are under study which are composed with a conjugate transposition operator. As our overview shows, a substantial part of problems known in the literature directly falls in this category or can be placed into it after some equivalent transformations.

In the description of the problems, we use the symbols \mathbf{A} , \mathbf{B} , and \mathbf{C} for given matrices, \mathbf{b} , \mathbf{d} , \mathbf{p} , and \mathbf{q} for given vectors, and \mathbf{x} for the unknown vector. Matrix and vector operations are considered as defined in terms of an appropriate idempotent semimodule. The minus in the exponent denotes the vector multiplicative conjugate transpose. Further details on the notation used here and throughout the paper, and related definition are given in the next section.

2.1 Linear objective functions

One of the long-known and well-studied tropical optimization problems is a direct tropical analog of linear programming problems, which is to

$$\begin{aligned} &\text{minimize} && \mathbf{p}^T \mathbf{x}, \\ &\text{subject to} && \mathbf{A}\mathbf{x} \geq \mathbf{d}. \end{aligned}$$

Complete closed-form solutions to the problem together with related duality results are given under various algebraic assumptions in a number of publications. Specifically, [16, 7] examine the problem in the case of general semirings; the former offers a solution based on an abstract extension of the conventional linear programming duality, and the second suggests a residual-based solution technique. Furthermore, [17] develops results in [16] to concentrate on the idempotent semifield $\mathbb{R}_{\max,+}$, whereas [33] offer a solution for $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$ in the context of the theory of max-separable functions.

A tropical optimization problem in $\mathbb{R}_{\max,+}$ with more constraints, which

can be written in the form

$$\begin{aligned} & \text{minimize} && \mathbf{p}^T \mathbf{x}, \\ & \text{subject to} && \mathbf{Ax} \leq \mathbf{d}, \quad \mathbf{Cx} \geq \mathbf{b}, \\ & && \mathbf{g} \leq \mathbf{x} \leq \mathbf{h}, \end{aligned}$$

is considered in [34, 35, 36, 26] within the framework of max-separable functions. Under quite general conditions, explicit solutions for the problem are given represented basically in conventional terms rather than in a closed form of tropical vector algebra.

For another problem in $\mathbb{R}_{\max,+}$ with two-sided equality constraints,

$$\begin{aligned} & \text{minimize} && \mathbf{p}^T \mathbf{x}, \\ & \text{subject to} && \mathbf{Ax} \oplus \mathbf{b} = \mathbf{Cx} \oplus \mathbf{d}, \end{aligned}$$

solutions are obtained in [18, 14, 27, 29]. In particular, [27] presents a pseudo-polynomial algorithm to evaluate a solution or to indicate that no solutions exist. The algorithm uses an alternating method offered by [37] that replaces the equality constraint with two opposite inequalities, which are solved by turns iteratively to provide progressively better bounds for a solution. A heuristic approach is suggested in [29] to get an approximate solution by use of local search techniques combined with iterative procedures for solving the problem in particular low-dimensional cases with one and two variables.

2.2 Nonlinear objective functions

We start with a particular problem that can be written in the form

$$\begin{aligned} & \text{minimize} && (\mathbf{Ax})^- \mathbf{d}, \\ & \text{subject to} && \mathbf{Ax} \leq \mathbf{d}. \end{aligned}$$

The problem is formulated as to find \mathbf{x} to provide a best underestimating approximation for \mathbf{d} by \mathbf{Ax} in the Chebyshev norm. It is examined in [15], where a complete explicit solution is given as an application of an abstract theory of linear operators over $\mathbb{R}_{\max,+}$ developed there. A similar solution is also suggested by [7].

Another two problems are initially represented in a different manner, but can be put in a form with tropical nonlinear objective function and linear constraints. A problem of minimizing a Chebyshev-like distance function under some constraints is defined in $\mathbb{R}_{\max,+}$ and solved with a polynomial-time threshold-type algorithm in [19]. In fact, with the same algebraic manipulations as in [23, 31], it can be rearranged to take a form with a nonlinear objective function and boundary constraints,

$$\begin{aligned} & \text{minimize} && (\mathbf{Ax})^- \mathbf{p} \oplus \mathbf{q}^- \mathbf{Ax}, \\ & \text{subject to} && \mathbf{g} \leq \mathbf{x} \leq \mathbf{h}. \end{aligned}$$

A problem of minimization of the range norm is considered in [28], where an explicit solution is suggested in a combined analytic framework that involves both the semifield $\mathbb{R}_{\max,+}$ and its dual semifield $\mathbb{R}_{\min,+}$. The problem can actually be written only in terms of $\mathbb{R}_{\max,+}$ as

$$\text{minimize } \mathbf{1}^T \mathbf{A} \mathbf{x} (\mathbf{A} \mathbf{x})^{-1},$$

where $\mathbf{1} = (1, \dots, 1)^T$ is a vector of ones (in terms of the semifield $\mathbb{R}_{\max,+}$).

On the basis of a connection between solutions of two-sided equality constraints and a mean payoff game established by [38], an iterative computational scheme is developed in [39] to find a solution in $\mathbb{R}_{\max,+}$ for a problem with nonlinear objective function given by

$$\begin{aligned} &\text{minimize } \mathbf{p}^T \mathbf{x} (\mathbf{q}^T \mathbf{x})^{-1}, \\ &\text{subject to } \mathbf{A} \mathbf{x} \oplus \mathbf{b} \leq \mathbf{C} \mathbf{x} \oplus \mathbf{d}. \end{aligned}$$

Now we consider problems that are formulated in terms of a general linear ordered radicable semifield and admit explicit solutions in a vector form. The analytic technique implemented to solve the problems is based on new results in tropical spectral theory and solutions of linear tropical inequality developed in [22, 25, 13].

First, a complete solution is given in [40] to an unconstrained problem

$$\text{minimize } \mathbf{x}^- \mathbf{A} \mathbf{x} \oplus \mathbf{x}^- \mathbf{p} \oplus \mathbf{q}^- \mathbf{x}.$$

Furthermore, [31] offers a direct explicit solution to another unconstrained problem in the form

$$\text{minimize } (\mathbf{A} \mathbf{x})^- \mathbf{p} \oplus \mathbf{q}^- \mathbf{A} \mathbf{x}.$$

It is also shown that two problems with linear inequality and equality constraints,

$$\begin{aligned} &\text{minimize } \mathbf{x}^- \mathbf{p} \oplus \mathbf{q}^- \mathbf{x}, & \text{minimize } \mathbf{x}^- \mathbf{p} \oplus \mathbf{q}^- \mathbf{x}, \\ &\text{subject to } \mathbf{A} \mathbf{x} \leq \mathbf{x}; & \text{subject to } \mathbf{A} \mathbf{x} = \mathbf{x}; \end{aligned}$$

reduce to the former unconstrained problem and so get explicit solutions.

Finally, in a brief conference paper [20], a complete direct solution is given to a problem

$$\begin{aligned} &\text{minimize } \mathbf{x}^- \mathbf{A} \mathbf{x}, \\ &\text{subject to } \mathbf{B} \mathbf{x} \leq \mathbf{x} \end{aligned}$$

under the condition that at least one of the matrices \mathbf{A} and \mathbf{B} is irreducible.

In this paper, we examine in detail a more general version of the last problem with arbitrary matrices and with additional constraints included.

3 Preliminary definitions and results

The purpose of this section is to offer a brief introduction to tropical (idempotent) mathematics so as to provide an appropriate theoretical framework for subsequent analysis and solution of tropical optimization problems.

Both concise introductions to and comprehensive presentations of the theory and methods are suggested in a range of published works, including [41, 5, 42, 6, 7, 8, 9, 10, 11, 43, 12, 14], which sometimes vary in the notation system adopted and in the form of results presented.

Below we give an overview of definitions, notation, and background results that mainly follow [25, 22, 13] and form a basis for complete solution of the problem under study in an explicit vector form as well as for clear geometric illustrations. Additional details and an extensive bibliography can be found in the publications listed above.

3.1 Idempotent semifield

Let \mathbb{X} be a set that is equipped with two operations: addition \oplus and multiplication \otimes , and contains their related neutral elements: null $\mathbb{0}$ and identity $\mathbb{1}$. Both operations are assumed to be associative and commutative, and multiplication to be distributive over addition. Furthermore, addition is idempotent providing $x \oplus x = x$ for all $x \in \mathbb{X}$. Multiplication is invertible, which means that there exist a multiplicative inverse x^{-1} for any $x \in \mathbb{X}_+$, where $\mathbb{X}_+ = \mathbb{X} \setminus \{\mathbb{0}\}$. Endowed with these properties, the algebraic structure $\langle \mathbb{X}, \mathbb{0}, \mathbb{1}, \oplus, \otimes \rangle$ is commonly referred to in the literature as the idempotent semifield.

Due to the associativity of multiplication, the power notation with integer exponents is introduced in the usual way. For any $x \in \mathbb{X}_+$ and an integer $p \geq 1$, it is defined as $x^0 = \mathbb{1}$, $\mathbb{0}^p = \mathbb{0}$, $x^p = x^{p-1} \otimes x = x \otimes x^{p-1}$, and $x^{-p} = (x^{-1})^p$. Moreover, we assume that the integer power extends to rational exponents, which makes the semifield radicable.

From here on, we leave out the multiplication sign \otimes for brevity sake, whereas the exponents are considered in the sense of the above notation.

There is a partial order induced on \mathbb{X} by idempotent addition so that $x \leq y$ if and only if $x \oplus y = y$. The definition implies an extremal property of addition in the form of inequalities $x \leq x \oplus y$ and $y \leq x \oplus y$. Moreover, according to the order, both addition and multiplication prove to be isotone.

Furthermore, the partial order is assumed to extend to a total order, which makes the semifield linearly ordered. Throughout the paper, we interpret the relation symbols and the problem formulations in terms of this linear order.

Instances of the linearly ordered radicable idempotent semifield include $\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, + \rangle$, $\mathbb{R}_{\min,+} = \langle \mathbb{R} \cup \{+\infty\}, +\infty, 0, \min, + \rangle$, $\mathbb{R}_{\max,\times} = \langle \mathbb{R}_+ \cup \{0\}, 0, 1, \max, \times \rangle$, and $\mathbb{R}_{\min,\times} = \langle \mathbb{R}_+ \cup \{+\infty\}, +\infty, 1, \min, \times \rangle$,

where \mathbb{R} is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$.

For example, the semifield $\mathbb{R}_{\max,+}$ contains the null $0 = -\infty$ and the identity $1 = 0$. There is an inverse x^{-1} defined for each $x \in \mathbb{R}$ and equal to $-x$ in conventional algebra. For any $x, y \in \mathbb{R}$, the power x^y coincides with the arithmetic product xy . The order induced by idempotent addition corresponds to the natural linear order on \mathbb{R} .

The semifield $\mathbb{R}_{\min,\times}$ is equipped with $0 = +\infty$ and $1 = 1$. The inverse and power notations have the standard interpretation. The relation \leq defines a reverse order to the linear order on \mathbb{R} .

Finally note that all above semifields are isomorphic to each other.

3.2 Idempotent semimodule

Consider \mathbb{X}^n to be a Cartesian power of \mathbb{X} with elements taken to be column vectors. There is a vector $0 \in \mathbb{X}^n$ that has all components equal to 0 . Any vector without zero components is called regular. The set of all regular vectors in \mathbb{X}^n is represented by \mathbb{X}_+^n .

For any vectors $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_i)$ from \mathbb{X}^n , and a scalar $x \in \mathbb{X}$, vector addition and scalar multiplication are defined component-wise according to the usual rules

$$\{\mathbf{a} \oplus \mathbf{b}\}_i = a_i \oplus b_i, \quad \{x\mathbf{a}\}_i = xa_i.$$

Endowed with these operations, the Cartesian power \mathbb{X}^n forms an idempotent semimodule over \mathbb{X} .

The extremal property of idempotent addition extends in the semimodule to component-wise vector inequalities $\mathbf{a} \leq \mathbf{a} \oplus \mathbf{b}$ and $\mathbf{b} \leq \mathbf{a} \oplus \mathbf{b}$. The isotone properties of scalar operations imply that both vector addition and scalar multiplication are also isotone in each argument.

To provide a clear visualization of key phenomena inherent in idempotent semimodules, we use graphical illustrations in terms of the semimodule $\mathbb{R}_{\max,+}^2$. Any vector in $\mathbb{R}_{\max,+}^2$ is shown on the plane with a Cartesian coordinate system that can be supplemented by two artificial points if needed. The points are respectively put before the left end of the horizontal axis and below the bottom of the vertical axis to indicate, in a schematic way, the value $0 = -\infty$ required to represent zero coordinates of irregular vectors.

A geometric interpretation of the operations in the context of the semimodule $\mathbb{R}_{\max,+}^2$ is given in Figures 1 and 2. Idempotent addition of two vectors with a common base point at the origin of the coordinate system goes by a “rectangle rule”. According to the rule the sum is obtained as the upper right vertex of an upright finite (Figure 1, left) or infinite (Figure 1, right) rectangle that has vertices at the end points of the vectors to add.

Scalar multiplication of a vector by a nonzero scalar results in moving the end point of the vector along a line at 45° to the coordinate axes for

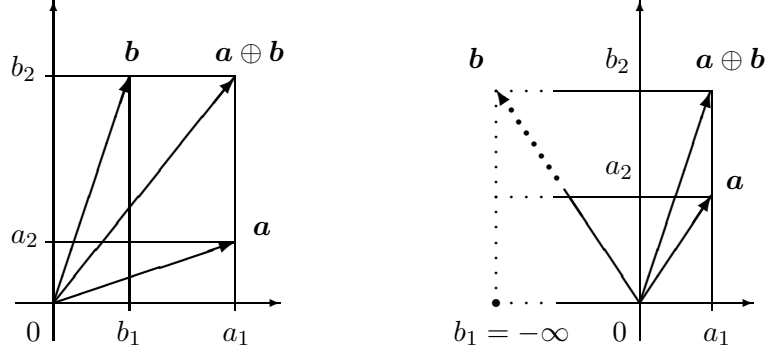


Figure 1: Vector addition of regular vectors (left) and of regular and irregular vectors (right).

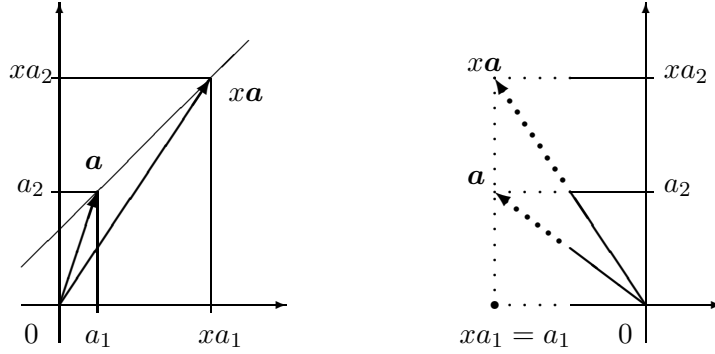


Figure 2: Scalar multiplication of a regular vector (left) and of an irregular vector (right).

regular vectors (Figure 2, left) or along an artificial vertical (horizontal) line for irregular vectors (Figure 2, right).

As usual, a vector \mathbf{b} is said to be linearly dependent on vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ if $\mathbf{b} = x_1 \mathbf{a}_1 \oplus \dots \oplus x_m \mathbf{a}_m$ for some scalars x_1, \dots, x_m . Specifically, two vectors \mathbf{a} and \mathbf{b} are collinear if there is a scalar x such that $\mathbf{b} = x\mathbf{a}$.

Given vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$, the linear combinations $x_1 \mathbf{a}_1 \oplus \dots \oplus x_m \mathbf{a}_m$ for all x_1, \dots, x_m form a linear span of the vectors and constitute an idempotent subsemimodule. Graphical examples of the linear span in $\mathbb{R}_{\max, +}^2$ are shown in Figure 3 as areas surrounded by hatched borders.

The linear span of two regular vectors is a strip bounded by the lines drawn through the end points of the vectors (Figure 3, left). When one of the vectors is irregular, the strip extends to take a form of a half-plane (Figure 3, right).

For any nonzero column vector $\mathbf{a} = (a_i)$, we define its multiplicative conjugate transpose as a row vector $\mathbf{a}^- = (a_i^-)$ with components $a_i^- = a_i^{-1}$.

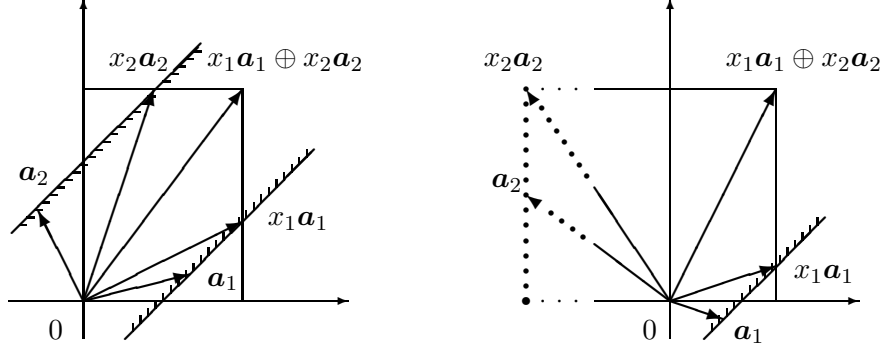


Figure 3: Linear span of regular vectors (left) and of regular and irregular vectors (right).

if $a_i > 0$, and $a_i^- = 0$ otherwise. If both vectors \mathbf{a} and \mathbf{b} are regular, then the component-wise inequality $\mathbf{a} \leq \mathbf{b}$ implies $\mathbf{a}^- \geq \mathbf{b}^-$.

3.3 Matrix algebra

Consider matrices with entries from \mathbb{X} . For conforming matrices $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$, and $\mathbf{C} = (c_{ij})$, and scalar $x \in \mathbb{X}$, matrix addition and multiplication together with scalar multiplication follow the element-wise formulas

$$\{\mathbf{A} \oplus \mathbf{B}\}_{ij} = a_{ij} \oplus b_{ij}, \quad \{\mathbf{AC}\}_{ij} = \bigoplus_k a_{ik} c_{kj}, \quad \{x\mathbf{A}\}_{ij} = xa_{ij}.$$

Specifically, multiplication of a matrix $\mathbf{A} = (a_{ij})$ by a vector $\mathbf{x} = (x_i)$ of order n gives a vector with components

$$\{\mathbf{Ax}\}_i = a_{i1}x_1 \oplus \cdots \oplus a_{in}x_n.$$

The matrix operations are element-wise isotone in each argument. For any matrices \mathbf{A} and \mathbf{B} of the same size, it holds $\mathbf{A} \leq \mathbf{A} \oplus \mathbf{B}$ and $\mathbf{B} \leq \mathbf{A} \oplus \mathbf{B}$.

A matrix that has only zero entries is the zero matrix represented by $\mathbf{0}$.

Consider square matrices of order n over \mathbb{X} and denote the set of the matrices by $\mathbb{X}^{n \times n}$. Any matrix having all off-diagonal entries equal to $\mathbf{0}$ is a diagonal matrix. A diagonal matrix with $\mathbf{1}$ along the diagonal is the identity matrix denoted by \mathbf{I} .

The matrix power is introduced in a regular way. For any matrix $\mathbf{A} \in \mathbb{X}^{n \times n}$ and integer $p \geq 1$, it is defined as $\mathbf{A}^0 = \mathbf{I}$, $\mathbf{A}^p = \mathbf{A}^{p-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{p-1}$.

If all entries of a matrix above or below the diagonal are zero, the matrix is triangular. A triangular matrix with zero diagonal entries is strictly triangular.

A matrix is reducible if simultaneous permutations of rows and columns can put it into a block-triangular normal form, and irreducible otherwise. The lower triangular normal form of a matrix \mathbf{A} is given by

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & & \mathbf{0} \\ \vdots & \vdots & \ddots & \\ \mathbf{A}_{s1} & \mathbf{A}_{s2} & \dots & \mathbf{A}_{ss} \end{pmatrix}, \quad (1)$$

where \mathbf{A}_{ii} is either irreducible or zero matrix of order n_i , whereas \mathbf{A}_{ij} is an arbitrary matrix of size $n_i \times n_j$ for all $i = 1, \dots, s$, $j < i$, and $n_1 + \dots + n_s = n$.

The trace of any matrix $\mathbf{A} = (a_{ij})$ is routinely defined as

$$\text{tr } \mathbf{A} = a_{11} \oplus \dots \oplus a_{nn}.$$

For any matrices \mathbf{A} and \mathbf{B} , and scalar x it holds

$$\text{tr}(\mathbf{A} \oplus \mathbf{B}) = \text{tr } \mathbf{A} \oplus \text{tr } \mathbf{B}, \quad \text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}), \quad \text{tr}(x\mathbf{A}) = x \text{tr}(\mathbf{A}).$$

Application of these properties leads, in particular, to a binomial identity for traces derived by [20] in the form

$$\text{tr}(\mathbf{A} \oplus \mathbf{B})^m = \bigoplus_{k=1}^m \bigoplus_{i_1 + \dots + i_k = m-k} \text{tr}(\mathbf{AB}^{i_1} \dots \mathbf{AB}^{i_k}) \oplus \text{tr } \mathbf{B}^m, \quad (2)$$

which is valid for any non-negative integer m .

As usual, a scalar λ is an eigenvalue of a matrix \mathbf{A} , if there exists a nonzero vector \mathbf{x} such that

$$\mathbf{Ax} = \lambda \mathbf{x}.$$

Every irreducible matrix has only one eigenvalue, whereas reducible matrices may possess several eigenvalues. The maximal eigenvalue (in the sense of the order on \mathbb{X}) is called the spectral radius of \mathbf{A} and directly calculated as

$$\lambda = \bigoplus_{m=1}^n \text{tr}^{1/m}(\mathbf{A}^m).$$

The spectral radius λ of any matrix \mathbf{A} offers a useful extremal property considered by [21], which holds that

$$\min \mathbf{x}^- \mathbf{Ax} = \lambda,$$

where the minimum is taken over all regular vectors \mathbf{x} .

4 Explicit solution to a linear inequality

Explicit results for optimization problems to be derived in the next section make well use of direct solutions to the following problem. Given a matrix $\mathbf{A} \in \mathbb{X}^{n \times n}$ and vector $\mathbf{b} \in \mathbb{X}^n$, find all vectors $\mathbf{x} \in \mathbb{X}_+^n$ to satisfy the inequality

$$\mathbf{A}\mathbf{x} \oplus \mathbf{b} \leq \mathbf{x}. \quad (3)$$

There is an equation with the same terms as in (3), which is frequently referred to as the non-homogeneous Bellman equation. The equation is examined in a number of works, including [44, 6, 7, 45, 12, 46, 8, 9], whereas solution to the inequality seems to receive less attention in the literature.

Complete solutions for the inequality with both irreducible and reducible matrices can be found in [25, 13], where they are obtained by reducing the inequality to an equation. Below we concentrate on regular solutions in the case of general reducible matrices and offer a direct proof to get an explicit solution in a slightly different and more compact form.

For any matrix \mathbf{A} , we define a function that is given by

$$\text{Tr}(\mathbf{A}) = \text{tr } \mathbf{A} \oplus \cdots \oplus \text{tr } \mathbf{A}^n,$$

and a star operator that takes \mathbf{A} to a matrix

$$\mathbf{A}^* = \mathbf{I} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}^{n-1}.$$

Both the function and the star operator act together in an inequality for matrix powers that represents, in somewhat different terms, a known result apparently first suggested by [44]. The statement, which is referred to below as the Carré inequality, holds that the condition $\text{Tr}(\mathbf{A}) \leq \mathbb{1}$ involves, for all integer $k \geq 0$, the inequality

$$\mathbf{A}^k \leq \mathbf{A}^*.$$

For irreducible matrices, a solution to (3) can be represented as follows.

Lemma 1. *Let \mathbf{x} be the general regular solution of inequality (3) with an irreducible matrix \mathbf{A} . Then the following statements hold:*

1. *If $\text{Tr}(\mathbf{A}) \leq \mathbb{1}$, then $\mathbf{x} = \mathbf{A}^*\mathbf{u}$ for all regular vectors $\mathbf{u} \geq \mathbf{b}$.*
2. *If $\text{Tr}(\mathbf{A}) > \mathbb{1}$, then there is no regular solution.*

Figure 4 offers examples of solutions to inequalities with a common matrix $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2)$ and different vectors \mathbf{b} in $\mathbb{R}_{\max,+}^2$. The set of solutions is indicated by semi-infinite regions with hatched borders. Together with the columns of the matrix \mathbf{A} , those of $\mathbf{A}^* = (\mathbf{a}_1^*, \mathbf{a}_2^*)$ are also presented.

In the rest of the section, we show that the result of Lemma 1 can directly be extended to solve the inequality with irreducible matrices. We start with an auxiliary result that provides a useful representation for $(\mathbf{A} \oplus \mathbf{B})^*$ and can be considered as an analogue of an identity that appears in [6].

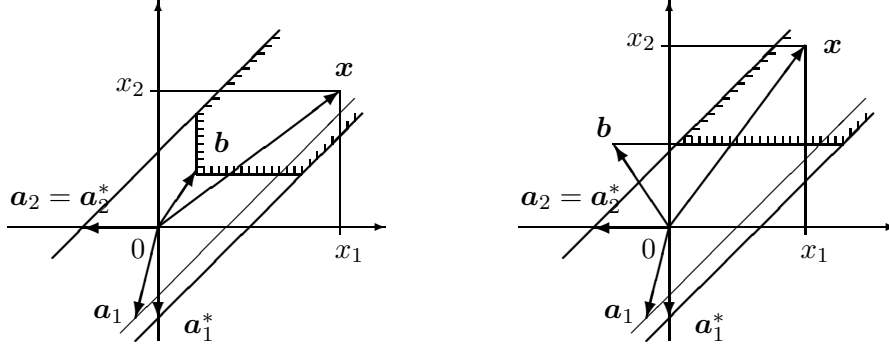


Figure 4: Solution to linear inequalities with an irreducible matrix.

Proposition 1. For any matrices A and B with $\text{Tr}(A \oplus B) \leq 1$, it holds

$$(A \oplus B)^* = (A^* B)^* A^*. \quad (4)$$

Proof. To verify the identity, it is sufficient to show that two opposite inequalities between its sides are valid. By expanding both left and right parts of (4) and rearranging the terms, we obtain one of the inequalities

$$\begin{aligned} (A \oplus B)^* &= \bigoplus_{m=0}^{n-1} \bigoplus_{0 \leq i_0 + i_1 + \dots + i_m \leq n-m-1} A^{i_0} (B A^{i_1}) \dots (B A^{i_m}) \\ &\leq \bigoplus_{m=0}^{n-1} \bigoplus_{0 \leq i_0, i_1, \dots, i_m \leq n-1} A^{i_0} (B A^{i_1}) \dots (B A^{i_m}) = (A^* B)^* A^*. \end{aligned}$$

Furthermore, we denote $C = A \oplus B$ and see that $A \leq C$ and $B \leq C$. Application of the Carré inequality gives

$$(A^* B)^* A^* \leq (C^* C)^* C^* = \bigoplus_{m=0}^{n^2-1} C^m = C^* = (A \oplus B)^*,$$

which shows that the opposite inequality is also true. \square

Now we are in a position to present the main result of the section.

Theorem 2. Let x be the general regular solution of inequality (3) with an arbitrary matrix A . Then the following statements hold:

1. If $\text{Tr}(A) \leq 1$, then $x = A^* u$ for all regular vectors $u \geq b$.
2. If $\text{Tr}(A) > 1$, then there is no regular solution.

Proof. If the matrix \mathbf{A} is irreducible, then the statement of the theorem is provided by Lemma 1. Suppose that \mathbf{A} is reducible. Provided that $\mathbf{A} = \mathbb{0}$, the theorem is trivially true. Consider the case when $\mathbf{A} \neq \mathbb{0}$ and assume, without loss of generality, the matrix \mathbf{A} to have the form of (1).

We define block-diagonal and strictly lower block-triangular matrices

$$\mathbf{D} = \begin{pmatrix} \mathbf{A}_{11} & & \mathbb{0} \\ & \ddots & \\ \mathbb{0} & & \mathbf{A}_{ss} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \mathbb{0} & \dots & \dots & \mathbb{0} \\ \mathbf{A}_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{A}_{s1} & \dots & \mathbf{A}_{s,s-1} & \mathbb{0} \end{pmatrix}$$

to be corresponding parts in the additive decomposition $\mathbf{A} = \mathbf{D} \oplus \mathbf{T}$.

According to the row partition of the matrix \mathbf{A} in the form (1), we write both vectors \mathbf{x} and \mathbf{b} in the block form

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_s \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_s \end{pmatrix},$$

where \mathbf{x}_i and \mathbf{b}_i are vectors of order n_i for each $i = 1, \dots, s$.

Furthermore, we represent (3) as a system of inequalities, one for each element of the partition,

$$\mathbf{A}_{i1}\mathbf{x}_1 \oplus \dots \oplus \mathbf{A}_{ii}\mathbf{x}_i \oplus \mathbf{b}_i \leq \mathbf{x}_i, \quad i = 1, \dots, s.$$

Suppose that $\text{Tr}(\mathbf{A}) \leq \mathbb{1}$. In this case, for each i , we have

$$\text{Tr}(\mathbf{A}_{ii}) \leq \text{Tr}(\mathbf{A}_{11}) \oplus \dots \oplus \text{Tr}(\mathbf{A}_{ss}) = \text{Tr}(\mathbf{A}) \leq \mathbb{1}.$$

It follows from Lemma 1 that inequality i can be solved with respect to \mathbf{x}_i . Assuming \mathbf{v}_i to be a regular vector of order n_i , the solution is given by

$$\mathbf{x}_i = \mathbf{A}_{ii}^* \mathbf{v}_i, \quad \mathbf{v}_i \geq \mathbf{A}_{i1}\mathbf{x}_1 \oplus \dots \oplus \mathbf{A}_{i,i-1}\mathbf{x}_i \oplus \mathbf{b}_i.$$

With another notation \mathbf{u}_i to represent a regular vector of order n_i , the last inequality can be rewritten as

$$\mathbf{v}_i = \mathbf{A}_{i1}\mathbf{x}_1 \oplus \dots \oplus \mathbf{A}_{i,i-1}\mathbf{x}_i \oplus \mathbf{u}_i, \quad \mathbf{u}_i \geq \mathbf{b}_i.$$

Turning back to the solution of inequality i , we arrive at the representation

$$\mathbf{x}_i = \mathbf{A}_{ii}^*(\mathbf{A}_{i1}\mathbf{x}_1 \oplus \dots \oplus \mathbf{A}_{i,i-1}\mathbf{x}_i) \oplus \mathbf{A}_{ii}^* \mathbf{u}_i, \quad \mathbf{u}_i \geq \mathbf{b}_i.$$

After concatenation of all vectors $\mathbf{u}_1, \dots, \mathbf{u}_s$ into one vector \mathbf{u} of order n , we write all solutions in the form of an implicit equation for \mathbf{x} ,

$$\mathbf{x} = \mathbf{D}^* \mathbf{T} \mathbf{x} \oplus \mathbf{D}^* \mathbf{u}, \quad \mathbf{u} \geq \mathbf{b}.$$

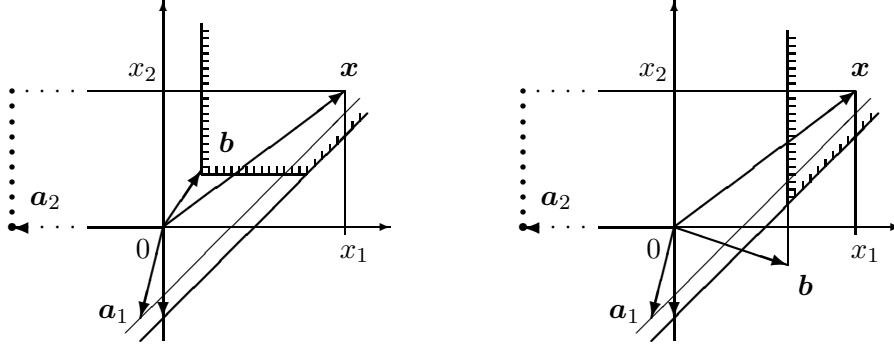


Figure 5: Solution to linear inequalities with a reducible matrix.

Considering a strictly block-triangular form of T and a conforming block-diagonal form of D^* , it is not difficult to verify that $(D^*T)^r = 0$ for all $r \geq s$.

By iterating the implicit equation, we arrive at a regular solution

$$x = (I \oplus (D^*T) \oplus \cdots \oplus (D^*T)^{s-1})D^*u = (D^*T)^*D^*u, \quad u \geq b.$$

With application of identity (4) to $A = D \oplus T$, we put the solution in the final form

$$x = A^*u, \quad u \geq b.$$

If $\text{Tr}(A) > 1$ then there is at least one row partition i with $\text{Tr}(A_{ii}) > 1$. In this case, according to Lemma 1, no regular solutions x_i exist for the partition and so no regular solutions x for (3). \square

A graphical illustration of solutions to the inequality with a common reducible matrix A in $\mathbb{R}_{\max,+}^2$ is given in Figure 5.

5 A constrained optimization problem

We start with a real-world problem drawn from project scheduling and then represent it in terms of the semifield $\mathbb{R}_{\max,+}$. The problem serves as motivation and illustration for the solution of a more common problem that is formulated in the context of a general idempotent semifield, which covers $\mathbb{R}_{\max,+}$ as a specific instance. As the main result, we obtain a solution to the common problem under fairly general assumptions. Particular cases of the problem are considered and illustrated with numerical and graphical examples in the framework of the semifield $\mathbb{R}_{\max,+}$.

5.1 A project scheduling problem

Below we concern with optimal scheduling of collection of activities (jobs, tasks), which presents a rather typical problem of schedule development in project management (see, e.g., [47, 48] for further details and examples).

Consider a project that involves n activities operating under precedence constraints, which are referred to as Start-to-Finish, Start-to-Start, and Early Start constraints. The Start-to-Finish constraints do not allow an activity to complete until predefined times after initiation of other activities. The activities are assumed to complete as early as possible to meet the constraints. The Start-to-Start constraints determine minimum allowed times between initiation of activities, whereas the Early Start constraints specify earliest possible dates for activity initiation.

Each activity in the project is characterized by its flow time (also known as turnaround or processing time) defined as the time interval between its initiation and completion. The problem is to design a schedule that minimizes the maximum flow time over all activities, subject to the above constraints.

For each activity $i = 1, \dots, n$, we denote its initiation time by x_i and its completion time by y_i . Let a_{ij} be a minimum possible time lag between initiation of activity $j = 1, \dots, n$ and completion of i .

Given a_{ij} , the completion time of activity i , due to the Start-to-Finish constraints, must satisfy the relations

$$x_j + a_{ij} \leq y_i, \quad j = 1, \dots, n,$$

where at least one inequality holds as an equality. Note that, in the case when a_{ij} is not given for some j , it is assumed to be $-\infty$.

The relations can be combined into one equality of the form

$$y_i = \max(x_1 + a_{i1}, \dots, x_n + a_{in}).$$

By replacing the ordinary operations with those in $\mathbb{R}_{\max,+}$, we get

$$y_i = a_{i1}x_1 \oplus \dots \oplus a_{in}x_n, \quad i = 1, \dots, n.$$

With a matrix and vectors,

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

we represent the scalar equalities in a vector form

$$\mathbf{y} = \mathbf{A}\mathbf{x}.$$

Furthermore, we formulate a problem of minimizing the maximum flow time over all activities in the project. In ordinary notation, the objective function in the problem is given by

$$\max(y_1 - x_1, \dots, y_n - x_n).$$

Since, in terms of $\mathbb{R}_{\max,+}$, the function is written as

$$x_1^{-1}y_1 \oplus \dots \oplus x_n^{-1}y_n = \mathbf{x}^- \mathbf{y} = \mathbf{x}^- \mathbf{A} \mathbf{x},$$

we arrive at a problem to find all regular \mathbf{x} so as to

$$\text{minimize } \mathbf{x}^- \mathbf{A} \mathbf{x}.$$

Now we incorporate Start-to-Start and Early Start constraints into the problem formulation. For each activity $i = 1, \dots, n$, we denote by g_i an early possible initiation time and by c_{ij} a minimum possible time lag between initiation of activity $j = 1, \dots, n$ and initiation of i . Given c_{ij} and g_i , the initiation time x_i for activity i is subject to the relations

$$\begin{aligned} \max(x_1 + c_{i1}, \dots, x_n + c_{in}) &\leq x_i, \\ g_i &\leq x_i. \end{aligned}$$

Representation in terms of $\mathbb{R}_{\max,+}$ gives scalar inequalities

$$\begin{aligned} c_{i1}x_1 \oplus \dots \oplus c_{in}x_n &\leq x_i, \\ g_i &\leq x_i. \end{aligned}$$

With the matrix-vector notation $\mathbf{C} = (c_{ij})$ and $\mathbf{g} = (g_i)$, we get

$$\begin{aligned} \mathbf{C} \mathbf{x} &\leq \mathbf{x}, \\ \mathbf{g} &\leq \mathbf{x}. \end{aligned}$$

The optimal scheduling problem under consideration takes the form

$$\begin{aligned} &\text{minimize } \mathbf{x}^- \mathbf{A} \mathbf{x}, \\ &\text{subject to } \mathbf{C} \mathbf{x} \leq \mathbf{x}, \\ &\mathbf{g} \leq \mathbf{x}. \end{aligned}$$

We note in conclusion that the solution of the problem (if exists) is not unique. Specifically, if there is a solution \mathbf{x} , then $\alpha \mathbf{x}$, where $\alpha > 0$, is also a solution. In the context of project scheduling, however, a unique solution naturally appears when the minimal solution is of interest.

5.2 A general representation

Let \mathbb{X} be a linearly ordered radicable idempotent semifield, $\mathbf{A}, \mathbf{C} \in \mathbb{X}^{n \times n}$ be given matrices and $\mathbf{g} \in \mathbb{X}^n$ be a given vector. The problem is to find all regular solutions $\mathbf{x} \in \mathbb{X}_+^n$ so as to

$$\begin{aligned} & \text{minimize} && \mathbf{x}^- \mathbf{A} \mathbf{x}, \\ & \text{subject to} && \mathbf{C} \mathbf{x} \leq \mathbf{x}, \\ & && \mathbf{g} \leq \mathbf{x}. \end{aligned} \tag{5}$$

Since, for all vectors \mathbf{x} , the inequalities $\mathbf{C} \mathbf{x} \leq \mathbf{x}$ and $\mathbf{x} \geq \mathbf{g}$ can readily be combined into one equivalent inequality $\mathbf{C} \mathbf{x} \oplus \mathbf{g} \leq \mathbf{x}$, the problem has an alternative representation in the form

$$\begin{aligned} & \text{minimize} && \mathbf{x}^- \mathbf{A} \mathbf{x}, \\ & \text{subject to} && \mathbf{C} \mathbf{x} \oplus \mathbf{g} \leq \mathbf{x}. \end{aligned} \tag{6}$$

Consider the set of regular vectors that are determined by the inequality constraints in the problem and find out when the set is nonempty. Clearly, an appropriate necessary and sufficient condition follows from Theorem 2 and requires that $\text{Tr}(\mathbf{C}) \leq \mathbb{1}$.

5.3 The main result

A complete solution to the optimization problem is given as follows.

Theorem 3. *Suppose \mathbf{A} is a matrix with spectral radius $\lambda > 0$, \mathbf{C} is a matrix with $\text{Tr}(\mathbf{C}) \leq \mathbb{1}$, and \mathbf{g} is a vector. Define a scalar*

$$\theta = \lambda + \bigoplus_{k=1}^{n-1} \bigoplus_{1 \leq i_1 + \dots + i_k \leq n-k} \text{tr}^{1/k}(\mathbf{A} \mathbf{C}^{i_1} \dots \mathbf{A} \mathbf{C}^{i_k}) \tag{7}$$

and a matrix

$$\mathbf{S} = \theta^{-1} \mathbf{A} \oplus \mathbf{C}. \tag{8}$$

Then the minimum in (5) is equal to θ and attained if and only if

$$\mathbf{x} = \mathbf{S}^* \mathbf{u}, \quad \mathbf{u} \geq \mathbf{g}. \tag{9}$$

Proof. We start with the problem in the form of (6) and then follow the approach suggested in [20]. We introduce an additional variable so as to reduce the problem to an inequality. Furthermore, existence conditions for solutions of the inequality is used to determine the variable, whereas the solution of the inequality is taken as the solution of the problem.

Suppose θ is the minimum of the objective function in the problem and note that $\theta \geq \lambda > 0$. All regular vectors \mathbf{x} that yield the minimum are therefore determined by the system

$$\begin{aligned}\mathbf{x}^- \mathbf{A} \mathbf{x} &= \theta, \\ \mathbf{C} \mathbf{x} \oplus \mathbf{g} &\leq \mathbf{x}.\end{aligned}$$

Consider the first equality $\mathbf{x}^- \mathbf{A} \mathbf{x} = \theta$. Since for all \mathbf{x} we actually have $\mathbf{x}^- \mathbf{A} \mathbf{x} \geq \theta$, the equality can be further replaced with an inequality $\mathbf{x}^- \mathbf{A} \mathbf{x} \leq \theta$.

Next we take the inequality $\mathbf{x}^- \mathbf{A} \mathbf{x} \leq \theta$ and multiply it by $\theta^{-1} \mathbf{x}$ from the left. Since $\mathbf{x} \mathbf{x}^- \geq \mathbf{I}$ for all regular \mathbf{x} , we have $\theta^{-1} \mathbf{A} \mathbf{x} \leq \theta^{-1} \mathbf{x} \mathbf{x}^- \mathbf{A} \mathbf{x} \leq \mathbf{x}$ and so get a new inequality $\theta^{-1} \mathbf{A} \mathbf{x} \leq \mathbf{x}$.

Considering that left multiplication of the last inequality by $\theta \mathbf{x}^-$ and an obvious equality $\mathbf{x}^- \mathbf{x} = \mathbb{1}$ give the former one, both inequalities are equivalent.

Now the solution set of the problem is given by the system of inequalities

$$\begin{aligned}\theta^{-1} \mathbf{A} \mathbf{x} &\leq \mathbf{x}, \\ \mathbf{C} \mathbf{x} \oplus \mathbf{g} &\leq \mathbf{x}.\end{aligned}$$

It is easy to verify that the system is equivalent to one inequality,

$$\mathbf{S} \mathbf{x} \oplus \mathbf{g} \leq \mathbf{x}, \tag{10}$$

where the matrix \mathbf{S} is defined as in (8).

It follows from Theorem 2 that a necessary and sufficient condition for regular solutions of the inequality to exist is given by

$$\text{Tr}(\mathbf{S}) = \text{Tr}(\theta^{-1} \mathbf{A} \oplus \mathbf{C}) \leq \mathbb{1}. \tag{11}$$

To further examine the existence condition, we rewrite the left side of the inequality and then apply binomial identity (2) to get

$$\begin{aligned}\text{Tr}(\mathbf{S}) &= \bigoplus_{m=1}^n \text{tr}(\theta^{-1} \mathbf{A} \oplus \mathbf{C})^m \\ &= \bigoplus_{m=1}^n \bigoplus_{k=1}^m \bigoplus_{i_1 + \dots + i_k = m-k} \theta^{-k} \text{tr}(\mathbf{A} \mathbf{C}^{i_1} \dots \mathbf{A} \mathbf{C}^{i_k}) \oplus \text{Tr}(\mathbf{C}).\end{aligned}$$

Furthermore, rearrangement of terms in the last sum gives the following expression

$$\text{Tr}(\mathbf{S}) = \bigoplus_{k=1}^n \bigoplus_{0 \leq i_1 + \dots + i_k \leq n-k} \theta^{-k} \text{tr}(\mathbf{A} \mathbf{C}^{i_1} \dots \mathbf{A} \mathbf{C}^{i_k}) \oplus \text{Tr}(\mathbf{C}).$$

Considering that $\text{Tr}(\mathbf{C}) \leq \mathbb{1}$ by the conditions of the theorem, existence condition (11) reduces to inequalities

$$\bigoplus_{0 \leq i_1 + \dots + i_k \leq n-k} \theta^{-k} \text{tr}(\mathbf{A}\mathbf{C}^{i_1} \dots \mathbf{A}\mathbf{C}^{i_k}) \leq \mathbb{1}, \quad k = 1, \dots, n.$$

By solving the inequalities with respect to θ , we arrive at inequalities

$$\theta \geq \bigoplus_{0 \leq i_1 + \dots + i_k \leq n-k} \text{tr}^{1/k}(\mathbf{A}\mathbf{C}^{i_1} \dots \mathbf{A}\mathbf{C}^{i_k}), \quad k = 1, \dots, n,$$

or, equivalently, at one inequality

$$\begin{aligned} \theta &\geq \bigoplus_{k=1}^n \bigoplus_{0 \leq i_1 + \dots + i_k \leq n-k} \text{tr}^{1/k}(\mathbf{A}\mathbf{C}^{i_1} \dots \mathbf{A}\mathbf{C}^{i_k}) \\ &= \lambda \oplus \bigoplus_{k=1}^{n-1} \bigoplus_{1 \leq i_1 + \dots + i_k \leq n-k} \text{tr}^{1/k}(\mathbf{A}\mathbf{C}^{i_1} \dots \mathbf{A}\mathbf{C}^{i_k}). \end{aligned}$$

In order for θ to be the minimum in the problem, the last inequality must be satisfied as an equality, which gives (7).

Finally, application of Theorem 2 to inequality (10) leads to the solution in the form of (9). \square

Note that the minimum solution of the problem is given by $\mathbf{x}_0 = \mathbf{S}^* \mathbf{g}$.

5.4 Particular cases

First assume that $\mathbf{C} = \mathbb{0}$ and consider a problem

$$\begin{aligned} &\text{minimize} \quad \mathbf{x}^- \mathbf{A} \mathbf{x}, \\ &\text{subject to} \quad \mathbf{x} \geq \mathbf{g}. \end{aligned} \tag{12}$$

The next solution to the problem is a direct consequence of Theorem 3.

Corollary 4. *Suppose \mathbf{A} is a matrix with spectral radius $\lambda > 0$ and \mathbf{g} is a vector. Then the minimum in (12) is equal to λ and attained if and only if*

$$\mathbf{x} = (\lambda^{-1} \mathbf{A})^* \mathbf{u}, \quad \mathbf{u} \geq \mathbf{g}.$$

Now consider the case when a problem has no lower bound constraints and so is given by

$$\begin{aligned} &\text{minimize} \quad \mathbf{x}^- \mathbf{A} \mathbf{x}, \\ &\text{subject to} \quad \mathbf{C} \mathbf{x} \leq \mathbf{x}. \end{aligned} \tag{13}$$

By setting $\mathbf{g} = \mathbb{0}$ in Theorem 3, we immediately get the following result.

Corollary 5. Suppose \mathbf{A} is a matrix with spectral radius $\lambda > 0$ and \mathbf{C} is a matrix with $\text{Tr}(\mathbf{C}) \leq 1$. Define a scalar θ by (7) and a matrix \mathbf{S} by (8). Then the minimum in (13) is equal to θ and attained if and only if

$$\mathbf{x} = \mathbf{S}^* \mathbf{u}, \quad \mathbf{u} > 0.$$

It is easy to see that, when at least one of the matrices \mathbf{A} and \mathbf{C} is irreducible, the last result coincides with that in [20].

Finally, for a problem without any linear constraint in the form

$$\text{minimize } \mathbf{x}^T \mathbf{A} \mathbf{x}, \tag{14}$$

the solution offered by the theorem reduces to the following.

Corollary 6. For any matrix \mathbf{A} with spectral radius $\lambda > 0$, the minimum in (14) is equal to λ and attained if and only if

$$\mathbf{x} = (\lambda^{-1} \mathbf{A})^* \mathbf{u}, \quad \mathbf{u} > 0.$$

Note that this result is quite consistent with that in [40].

5.5 Illustrative examples

To illustrate the results obtained for problem (5) with both irreducible and reducible matrices, we give numerical and graphical examples in $\mathbb{R}_{\max, +}^2$.

5.5.1 Irreducible matrices

We start with a problem with irreducible matrices \mathbf{A} and \mathbf{C} , which are defined, together with a vector \mathbf{g} , as follows:

$$\mathbf{A} = \begin{pmatrix} 0 & -2 \\ -7 & -3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & -10 \\ 4 & -3 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} -9 \\ 6 \end{pmatrix}.$$

First consider problem (14), which has no constraints. Taking into account the matrices \mathbf{A} and

$$\mathbf{A}^2 = \begin{pmatrix} 0 & -2 \\ -7 & -3 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ -7 & -3 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -7 & -6 \end{pmatrix},$$

we find the spectral radius $\lambda = \text{tr } \mathbf{A} \oplus \text{tr}^{1/2}(\mathbf{A}^2) = 0 = 1$. Then we get

$$(\lambda^{-1} \mathbf{A})^* = \mathbf{A}^* = \mathbf{I} \oplus \mathbf{A} = \begin{pmatrix} 0 & -2 \\ -7 & 0 \end{pmatrix}.$$

The solution to the unconstrained problem takes the form

$$\mathbf{x} = \begin{pmatrix} 0 & -2 \\ -7 & 0 \end{pmatrix} \mathbf{u}, \quad \mathbf{u} \in \mathbb{R}^2,$$

which is graphically represented as in Figure 6 (left).

Furthermore, we obtain the solution to the inequality constraints in the form $\mathbf{C}\mathbf{x} \oplus \mathbf{g} \leq \mathbf{x}$. To apply Theorem 2, we take the matrices \mathbf{C} and

$$\mathbf{C}^2 = \begin{pmatrix} 0 & -10 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 0 & -10 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 0 & -10 \\ 4 & -6 \end{pmatrix}.$$

Then we verify that $\text{Tr}(\mathbf{C}) = \text{tr } \mathbf{C} \oplus \text{tr } \mathbf{C}^2 = 0 = \mathbb{1}$. After calculating

$$\mathbf{C}^* = \mathbf{I} \oplus \mathbf{C} = \begin{pmatrix} 0 & -10 \\ 4 & 0 \end{pmatrix}$$

we arrive at the solution (see Figure 6, right)

$$\mathbf{x} = \begin{pmatrix} 0 & -10 \\ 4 & 0 \end{pmatrix} \mathbf{u}, \quad \mathbf{u} \geq \begin{pmatrix} -9 \\ 6 \end{pmatrix}.$$

Now we use Theorem 3 to handle the constrained problem. We calculate

$$\mathbf{A}\mathbf{C} = \begin{pmatrix} 0 & -2 \\ -7 & -3 \end{pmatrix} \begin{pmatrix} 0 & -10 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -6 \end{pmatrix},$$

and then get $\theta = \lambda \oplus \text{tr}(\mathbf{A}\mathbf{C}) = 2$. Furthermore, we have

$$\mathbf{S} = \theta^{-1} \mathbf{A} \oplus \mathbf{C} = \begin{pmatrix} 0 & -4 \\ 4 & -3 \end{pmatrix}, \quad \mathbf{S}^* = \mathbf{I} \oplus \mathbf{S} = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}.$$

All solutions to the problem take the form

$$\mathbf{x} = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} \mathbf{u}, \quad \mathbf{u} \geq \begin{pmatrix} -9 \\ 6 \end{pmatrix}.$$

To simplify the solution, note that the columns in the matrix \mathbf{S}^* are collinear. Therefore, we can write

$$\mathbf{x} = \mathbf{S}^* \mathbf{u} = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \begin{pmatrix} 0 & -4 \end{pmatrix} \mathbf{u}.$$

With a new scalar variable

$$v = \begin{pmatrix} 0 & -4 \end{pmatrix} \mathbf{u} \geq \begin{pmatrix} 0 & -4 \end{pmatrix} \mathbf{g} = \begin{pmatrix} 0 & -4 \end{pmatrix} \begin{pmatrix} -9 \\ 6 \end{pmatrix} = 2,$$

the solution of problem (5) gets its final form

$$\mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} v, \quad v \geq 2.$$

Figure 6 (right) combines the solutions constraints with the final solution of the entire constrained problem. The final solution is depicted with a thick half-line that coincides with the right border of the feasible area defined by the constraints. The half-line starts at the end point of the vector \mathbf{x}_0 representing the minimal solution.

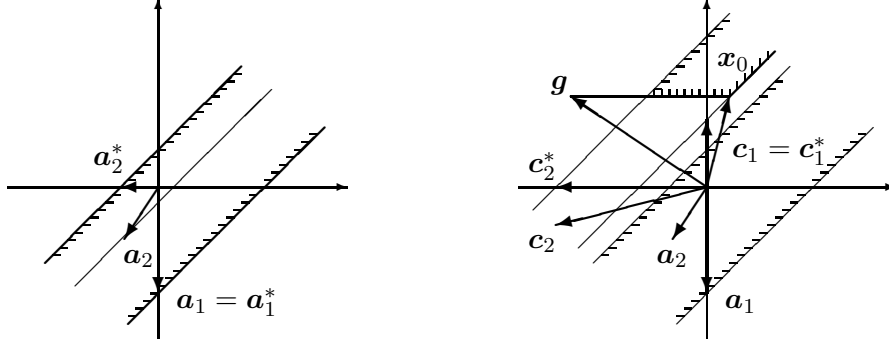


Figure 6: Solutions to unconstrained (left) and constrained (right) problems with irreducible matrices.

5.5.2 Reducible matrices

Now we solve problem (5) with reducible matrices under the conditions that

$$A = \begin{pmatrix} -2 & -\infty \\ -4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -6 \\ -\infty & -4 \end{pmatrix}, \quad g = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

We successively calculate

$$A^2 = \begin{pmatrix} -4 & -\infty \\ -4 & 0 \end{pmatrix}, \quad \lambda = 0, \quad (\lambda^{-1}A)^* = A^* = \begin{pmatrix} 0 & -\infty \\ -4 & 0 \end{pmatrix}.$$

The solution to the problem without constraints is given by (see Figure 7, left)

$$x = \begin{pmatrix} 0 & -\infty \\ -4 & 0 \end{pmatrix} u, \quad u \in \mathbb{R}^2.$$

Furthermore, we obtain the solution to the inequality constraints. Since

$$C^2 = \begin{pmatrix} 0 & -6 \\ -\infty & -8 \end{pmatrix}, \quad \text{Tr}(C) = 0, \quad C^* = \begin{pmatrix} 0 & -6 \\ -\infty & 0 \end{pmatrix},$$

the solution takes the form (Figure 7, right)

$$x = \begin{pmatrix} 0 & -6 \\ -\infty & 0 \end{pmatrix} u, \quad u \geq \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

To solve the constrained problem, we apply Theorem 3. We get

$$AC = \begin{pmatrix} -2 & -8 \\ -4 & -4 \end{pmatrix}, \quad \theta = 0, \quad S = S^* = \begin{pmatrix} 0 & -6 \\ -4 & 0 \end{pmatrix},$$

and then write the solution of problem (5) in the form

$$x = \begin{pmatrix} 0 & -6 \\ -4 & 0 \end{pmatrix} u, \quad u \geq \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

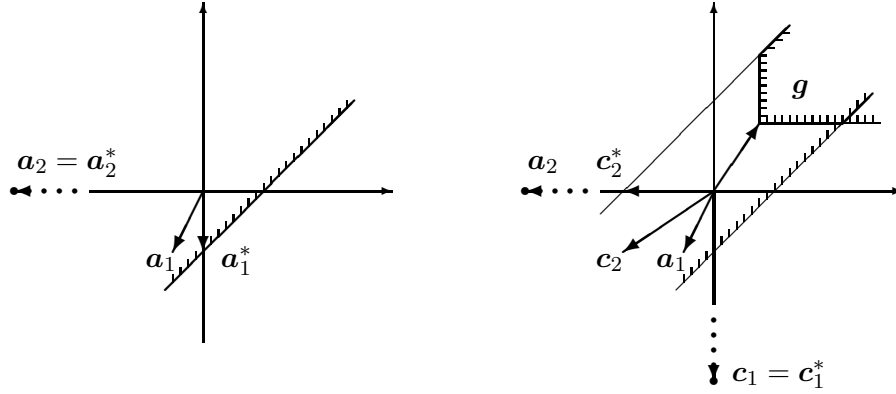


Figure 7: Solutions to unconstrained (left) and constrained (right) problems with reducible matrices.

The solution is shown in Figure 7 (right) as a half-strip region surrounded by a thick hatched border.

6 Conclusions

We started this paper with an overview of known tropical optimization problems, which demonstrates that tropical optimization is currently under intense development as a promising solution approach for many real-world problems in operations research. The paper examines a new optimization problem with a nonlinear objective function and linear inequality constraints in a rather general setting. The problem sufficiently extends that in [20] by eliminating irreducibility conditions for matrices involved as well as by introducing additional constraints. We offer an example problem drawn from project scheduling as a real motivation and clear illustration of the study.

The derivation of the solution is based on definitions, notations, and basic results developed in [25, 22, 13], which are outlined in a brief introduction supplemented by appropriate graphical illustrations. As a preliminary step, a new complete solution to a linear inequality was obtained for the case of arbitrary matrices. The solution extends known results in [25, 13] and is of independent interest.

The solution to the optimization problem under study follows the approach suggested in [20], which reduces the problem to the solution of a linear inequality with a parametrized matrix. By applying the above result for linear inequalities, we obtain a complete solution to the optimization problem in a compact vector form. To illustrate the result, numerical solutions and graphical representations are given for two-dimensional problems.

Future research is expected to concentrate on further extension of the problem intended to account for new types of constraints, including equality

constraints, and more general form of the objective function. Another line of investigation will be development of new real-world applications of the results.

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